# Prescribed Time Spacecraft Attitude Control Using Time-Varying State-Feedback

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**A time-varying feedback control law based on backstepping is proposed for prescribed arbitrary time stabilization and tracking for a chain of integrators. Prescribed arbitrary time control permits the convergence time to be chosen regardless of the initial conditions or system parameters. The sufficient condition is used to demonstrate Lyapunov stability for the system. Analytical results and explanations are provided for the influence of convergence time on maximum needed control torque. Sliding mode control is paired with the time-varying feedback approach to make the control law robust to reject the bounded disturbance. The control law is then applied to a spacecraft's attitude motion that is susceptible to bounded disturbance. A quaternion-based attitude representation is utilized to avoid mathematical singularity. The control law is written using a full quaternion with four components instead of only three in the vector part. Finally, the results of numerical simulation are presented.**

## I. Nomenclature

| | | |
|---|---|---|
| $V$ | = | Lyapunov function |
| $q$ | = | quaternion |
| $\boldsymbol{\omega}$ | = | angular velocity quaternion |
| $\omega$ | = | angular velocity vector |
| $u$ | = | control input |
| $s$ | = | sliding surface |
| $J$ | = | inertia matrix |

## II. Introduction

In control system applications, a controlled system's convergence time is a crucial performance criterion. Achieving finite-time stability rather than asymptotic stability is a common goal in a variety of applications, such as spacecraft attitude control, missile guidance, hybrid formation flying and group consensus, as well as in online differentiators and state observers [1]. Many space operations, such as surveillance, spacecraft rendezvous and docking, and formation flying, have difficulties with attitude control subsystem. The amount of time it takes for the spacecraft to converge to the correct orientation is crucial since it might decide whether the mission succeeds. In practice, the attitude control system demands specified settling period, or when the system's states converge to equilibrium with a minimum steady state error [2]. However, since the attitude motion of spacecraft is inherently nonlinear, model parameters are sometimes unknown, and also, due to the presence of external disturbances, designing the attitude controller for high control accuracy and fast response in the prescribed settling time remains a challenging problem [3].

Recent research has focused on finite and fixed time control to address the problem of pre-determined settling time. Finite-time stability refers to the stability in the sense of Lyapunov such that its trajectories converge to zero in finite time. Sliding mode control is a widely used method for finite-time convergence. Fractional power feedback and homogeneity property are two popular approaches in sliding mode control for finite-time convergence and robustness against bounded disturbances. It should be noted that a fractional power feedback renders a discontinuous control action [4]. The finite-time stability of homogeneous systems is examined by [5] and it concludes that a homogeneous system is finite-time stable if and only if it is asymptotically stable and has a negative degree of homogeneity. The work in [6] defines finite-time stability and provides a meticulous analysis on finite-time stability of autonomous systems. The

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author in [7] has further investigated the homogeneity properties of higher order sliding modes and proved finite-time convergence of the same. Further, the work in [8] presents a state-feedback controller that provides global finite-time stability to a class of nonlinear systems that are dominated by lower triangular system using nonsmooth feedback. The authors in [9] addressed sufficient and necessary conditions for finite-time stability of affine systems having the uniqueness of solution in forward time without assuming the continuity of settling time function at origin. Controllers designed in [10] approaches to equilibrium in finite-time using non-singular terminal sliding mode control. The authors in [11] propose a non-singular terminal sliding mode control for a second order nonlinear plant such that it ensures a bound on settling time of state after reaching the sliding surface. Further, the authors in [3] proposes a novel integral sliding mode surface with the aid of the saturation function and a homogeneous theory, which can converge the system state to the equilibrium point in finite time.

The notion of fixed-time stability was introduced by [12] which is to control the global settling time which does not depend on initial conditions and system parameters and proposed two approaches of achieving guaranteed settling-time using polynomial feedback laws and modified second order sliding mode control. It should be emphasized that fixed-time control approach guarantees the boundedness of settling time but it does not allow the assignment of prescribed settling time within physically allowable range due to some restrictions. The author in [13] concludes that any finite-time convergent homogeneous sliding mode controller can be transformed into a fixed-time convergent with an upper bound on convergence time, which is independent from initial conditions. An implicit Lyapunov function-based approach for fixed and finite-time stabilization and linear matrix inequalities are used to define fixed and finite-time stability conditions which ultimately facilitates the tuning of controller for given settling-time was proposed by [14]. However, [11, 12, 14] ensures finite and fixed-time stability but they do not offer any direct and simple solution to controller tuning for a given settling time.

Then, a new class of systems called prescribed time stable came into existence by the the work of [15] where the authors explicitly take the settling time into account in its time-varying feedback control law to achieve prescribed finite-time regulation with no dependence on initial conditions and system parameters using a scaling function that grows unbounded towards the terminal time. Further, the paper [16] introduced the notion of arbitrary time stability and prescribed arbitrary time stability. The control law offered by [16] is based on a non-autonomous differential equation whose solution approaches to zero in a predefined settling-time. The paper [17] further extends the approach introduced by [16] and offers its integration with general sliding mode control. The authors in [18] applies the work of [16] on rigid body motion on SO(3). However, the work in [19] pointed out an error in stability analysis of [16] for second and higher order systems, which then was addressed by [20]. The paper [21] also has pointed out an important technical error in [16] that the claim of bounded input even if the state grows large was false. According to work in [21], input will be unacceptably large at initial time for large negative values of $x < 0$. This issue of large input specially at the initial time moment is also highlighted in [22] with simulation. Our work is motivated by the modification in the non-autonomous differential inequality offered by [16] such that the modified differential inequality always gives lesser control input.

Considering the aforementioned discussions, in this paper, a novel non-autonomous differential equation having time-varying gain is designed as a control input to ensure that system states settle to zero within a specified settling period, regardless of initial conditions or system parameters. It should be emphasized that in classical finite-time control, the settling time is defined by the initial condition of the system as well as a number of design considerations, and as a result, it is impossible to arbitrarily prescribe the settling time. The time-varying gain–based prescribed-time control uses regular state feedback, rendering a smooth control input. Control design and stability analysis for high-order systems has been simplified by using proposed prescribed-time control methodology, which depends only on normal Lyapunov differential inequality stability analysis rather than a fractional Lyapunov differential inequality. The controller parameters can be changed at will to change the rate of convergence, but they have no effect on the upper bound of the settling time. The boundedness of the control input for a second order system is demonstrated using the extreme value theorem. Analytical results and explanations are provided for the influence of convergence time on maximum needed control torque. The control scheme presented in this research ensures prescribed arbitrary time stability and tracking of spacecraft attitude motion employing quaternion attitude description. First, a general prescribed arbitrary time controller for a chain of arbitrary order integrators is obtained, and then, a controller for spacecraft attitude motion without external disturbance is derived. In addition to prescribed arbitrary time control, a sliding mode control employing exponential reaching law is used to reject the bounded disturbance. Other types of sliding surfaces can be employed depending on the requirements, but in the interest of keeping this control system simple, we utilized a linear sliding surface. Finally, numerical simulation results are shown to demonstrate the effectiveness of the suggested control strategy.

The following is a breakdown of how the paper remaining paper is structured: The dynamics and kinematics of spacecraft motion are described in section 3 and in section 4, preliminary definitions and the main results of prescribed arbitrary time stability are presented. Section 5 presents a generic control law derivation process up to 2nd order, which is then generalized for $n^{th}$ order systems, after which a control for spacecraft dynamics is derived and its stability is demonstrated. The sliding mode controller is integrated with the prescribed arbitrary time control law in the concluding portion of section 5. The simulation results are provided in section 6. Finally, section 7 concludes the paper.

## III. Spacecraft Attitude Motion

In this section, the rotational motion of a fully actuated spacecraft is considered and attitude is described in terms of quaternion. For deriving the motion dynamics, two coordinate frames are considered and also, it is assumed that the satellite is a rigid body. The reference frames are the Earth-centered inertial frame $(XYZ)$ and satellite body-fixed frame $(xyz)$ that coincides at the $c.g.$ of the satellite. Let the current attitude of the spacecraft $w.r.t.$ to inertial frame is represented by $q \in [q_0, q_1, q_2, q_3]^T$, where $q_v = [q_1, q_2, q_3]^T \in \mathbb{R}^3$ is the vector part and $q_0$ is the scalar part of quaternion and satisfies $q^T q = 1$. The governing equations of the attitude motion are given as follows

$$\dot{q} = \frac{1}{2} q \otimes \boldsymbol{\omega} \tag{1}$$

$$J\dot{\omega} = -\hat{\omega} J \omega + u + d(t) \tag{2}$$

where $\boldsymbol{\omega} = [0, \omega^T]^T \in \mathbb{R}^4$ is the angular velocity quaternion, $\omega = [\omega_1, \omega_2, \omega_3]^T \in \mathbb{R}^3$ denotes the angular velocity of the spacecraft in the body frame $(xyz)$ w.r.t. inertial frame $(XYZ)$, $J \in \mathbb{R}^{3\times3}$ is the moment of inertia tensor, $u \in \mathbb{R}^3$ denotes the control command and $d(t) \in \mathbb{R}^3$ denotes the external disturbance which is assumed to be bounded in this paper, $\hat{\omega} \in \mathbb{R}^{3\times3}$ is the skew-symmetric matrix that facilitates the cross product with $\omega \in \mathbb{R}^3$ as given by

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{3}$$

Let $q_d = [q_{0_d}, q_{1_d}, q_{2_d}, q_{3_d}]^T \in \mathbb{R}^4$ be the desired quaternion attitude and $q_d^{-1} = [q_{0_d}, -q_{1_d}, -q_{2_d}, -q_{3_d}]^T \in \mathbb{R}^4$ is the conjugate of $q_d$, then the quaternion error, denoted by $q_e \in \mathbb{R}^4$, can be written as [23]

$$q_e = q_d^{-1} \otimes q \tag{4}$$

And, the error kinematics is derived as

$$\dot{q_e} = q_d^{-1} \otimes (\dot{q} - \dot{q_d} \otimes q_e) \tag{5}$$

Using Eq. (1) for both $q_e$ and $q_d$, Eq. (5) becomes

$$\begin{aligned} \dot{q_e} &= \frac{1}{2} q_d^{-1} \otimes (\frac{1}{2} q \otimes \boldsymbol{\omega} - \frac{1}{2} q_d \otimes \boldsymbol{\omega}_d^D \otimes q_e) \\ &= \frac{1}{2} q_e \otimes (\boldsymbol{\omega} - q_e^{-1} \otimes \boldsymbol{\omega}_d^D \otimes q_e) \end{aligned} \tag{6}$$

Here, $\omega_d^D \in \mathbb{R}^4$ is the desired angular velocity quaternion of desired reference frame $(x_D y_D z_D)$ . Let $\boldsymbol{\omega}_d^B$ be the desired angular velocity quaternion of the body-frame $(xyz)$, then the transformation between angular velocities in body-frame and angular velocity in the desired frame can be made through the following expression

$$\boldsymbol{\omega}_d^D = q_e \otimes \boldsymbol{\omega}_d^B \otimes q_e^{-1}$$

As a result, Eq. (6) becomes

$$\dot{q_e} = \frac{1}{2} q_e \otimes (\boldsymbol{\omega} - \boldsymbol{\omega}_d^B) \tag{7}$$

By writing Eq. (7) as $\boldsymbol{\omega} - \boldsymbol{\omega}_{\boldsymbol{d}}^{\boldsymbol{B}} = 2 q_e^{-1} \otimes \dot{q_e}$, we get

$$\dot{\omega} - \dot{\omega}_d^B = 2\dot{q}_e^{-1} \otimes \dot{q}_e + 2q_e^{-1} \otimes \ddot{q}_e$$
$$= 2(\| q_e \|^2, 0) + 2q_e^{-1} \otimes \ddot{q}_e \tag{8}$$

The vector part of Eq. (8) can be written as

$$\dot{\omega} - \dot{\omega}_d^B = 2G\ddot{q}_e \tag{9}$$

By rearranging Eq. (9), we get

$$\ddot{q}_e = \frac{1}{2}G^T(\dot{\omega} - \dot{\omega}_d^B) \tag{10}$$

where

$$G = \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix}_e$$

It is to be noted that for a tracking problem, $\dot{\omega}_d^B$ will be non-zero and it is required to compute correctly in the body-frame $(xyz)$. Let $R(q_e)$ be a rotation matrix that transforms a vector from body-frame to desired frame, $\omega_d^B = R(q_e)^T\omega_d^D$. or

$$\dot{\omega}_d^B = \dot{R}(q_e)^T\omega_d^D + R(q_e)^T\dot{\omega}_d^D \tag{11}$$

In the following Section, the preliminaries definitions are briefly presented for defining some terminology associated with the control methodology that is discussed in Section (4)

## IV. Preliminary definitions and Stability Analysis

Consider the following nonlinear non-autonomous system

$$\dot{x} = g(t, x; \theta), x(t_0) = x_0 \tag{12}$$

where $x \in \mathbb{R}^n$ is system state, $\theta \in R^n$ is the time-invariant system parameter and $g : \mathbb{R}_+ \cup \{0\} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a non-linear function whose origin lies at $x_e = 0$, such that it satisfies $g(t, 0, \theta) = 0$ and the initial time is represented by $t_0 \in \mathbb{R}_+ \cup \{0\}$. The following definitions are used to obtain the prescribed time control that presented in the next section.
*Definition 1:*(Globally finite-time stable) The equilibrium point of system (12) is called globally finite-time stable if is globally asymptotically stable and any solution $x(t, x_0, \theta)$ of (12) converges to equilibrium point in some finite time moment. Alternatively, $x(t, x_0, \theta) = 0, \forall t \geq T(x_0, \theta)$, where $T(x_0, \theta)$ is settling time function which depends on initial conditions and system parameters [12].
*Definition 2:*(Fixed time stable) The equilibrium point of system (12) is called fixed time stable if it is globally finite time stable and $T(x_0, \theta)$ is upper bounded. Alternatively, $\exists T_{max} > 0 : T(x_0, \theta) \leq T_{max}$[12].
*Definition 3:*(Arbitrary time stable) The equilibrium point of the system (12) is said to be Arbitrary time stable if it is fixed time stable and $\exists T(x_0, \theta) > 0$ for some given $x_0$ and $\theta$, also if $T(x_0, \theta)$ can be adjusted arbitrarily by changing $\theta$ such that it possible to establish either of the two conditions

1. $T(x_0, \theta) \geq T_f$
2. $T(x_0, \theta) = T_f$ ,where $T_f$ is the true fixed time [16].

*Definition 4:* (Prescribed Arbitrary time stable) The equilibrium point of the system (12) is said to be Prescribed arbitrary time stable if it is fixed time stable and $\exists T > 0$ independent of $x_0$ and $\theta$ which can be chosen in advance and it is possible to establish either of the two conditions

1. $T \geq T_{tf}$ (Weak stable)
2. $T = T_{tf}$ (Strong stable) [16]

Based on the above definitions (1-3), the following remarks 1 and 2 are given:
*Remark 1:* Subject to definition 1, the states reach to the origin in finite-time based on the values of initial conditions and system parameters chosen. But, in the case of fixed-time control, the convergence time cannot be greater than $T_{max}$ which is known to the user in advance. Moreover, in fixed-time control, there is no means to set $T_{max}$ to arbitrary values.

This limitation can be handled in arbitrary time control, where the user can set the convergence time ($T(x_0, \theta)$) to any arbitrary value with adjusting $\theta$; but, it is not always permitted to change $\theta$. Hence, these limit the applicability of above strategies to the systems when desired convergence time is required.
*Remark 2:* In the evidence of above Remark 1, the definition 4 establishes prescribed time stable control which uses an uneven exponential transformation of system states with time varying feedback gain to achieve prescribe time control. Moreover, due to the use of uneven exponential transformation, the feedback gain becomes very high or more control effort is required while system states are negative (Remark 5). This limitation is adjusted in Proposition 1 with considering linear time varying state-feedback system. This resolves the issues of high gain requirements for the system to be controlled.
***Proposition 1:*** Consider the equilibrium point ($x_e$) of the system given by Eq. (12) lies in the domain $D \subset \mathbb{R}^n$. Assume that $\beta_1(x)$ and $\beta_2(x)$ are the positive definite continuous functions on $D \subset \mathbb{R}^n$, then $V{:}I \times D \to \mathbb{R}_+ \cup \{0\}$ is a real and continuously differentiable function that satisfies the following conditions

$$\beta_1(x) \le V(t,x) \le \beta_2(x), \forall t \in L, \forall x \in D - \{0\} \tag{13}$$

$$V(t,0) = 0, \forall t \in L \tag{14}$$

where $L = [t_0, t_f)$ and $\eta \ge 1$ is a real number, then it implies that the equilibrium point ($x_e = 0$) is
(I) weakly prescribed arbitrary time stable and $T_f \le t_f - t_0$, where $T_f$ is the real fixed time and $t_f$ is prescribed settling time, when the following inequality holds

$$\dot{V} \le \frac{-\eta V}{t_f - t}, \forall V \ne 0, \forall t \in L \tag{15}$$

(II) strongly prescribed arbitrary time stable and and $T_f = t_f - t_0$, when the following equality holds

$$\dot{V} = \frac{\eta V}{t_f - t}, \forall V \ne 0, \forall t \in L \tag{16}$$

Proof: Firstly, the uniform stability of the system in Eq. (12) is proved as a result of uniform stability and boundedness of the solution of $V(t,x)$ can be concluded. It can be easily shown from Eq. 15 that $\dot{V}(t,x) \le 0$ $\forall x$ and $t \in L$. Let $\lambda > 0$ and $\alpha > 0$ such that $B_\lambda \subset D$,$\alpha < min_{\|x\|=\lambda}\beta_1(x)$, then, $\{x \in B_\lambda \mid \beta_1(x) \le \alpha\}$ is the interior of $B_\lambda$ as shown in Fig. (1). Let's define a time-dependent set $\psi_{t,\alpha} = \{x \in B_\lambda | V(t,x) \le \alpha\}$. $\psi_{t,\alpha}$ contains $\{x \in B_\lambda | \beta_2(x) \le \alpha\}$ since $\beta_2(x) \le \alpha \implies V(t,x) \le \alpha$. On the other hand, $\psi_{t_0,\alpha}$ is a subset of $\{x \in B_\lambda | \beta_1(x) \le \alpha\}$. Since $V(t,x) \le \alpha \implies \beta_1(x) \le \alpha$,$\{x \in B_\lambda | \beta_2(x) \le \alpha\} \subset \psi_{t,\alpha} \subset \{x \in B_\lambda | \beta_2(x) \le \alpha\} \subset B_\lambda \subset D$ for all $t \in L$. Since $\dot{V}(t,x) \le 0$ on $D$ $\forall$ $x \in \psi_{t_0,\alpha}$ and $t \in L$, the solution starting at $(t_0, x_0)$ stays in $\psi_{t,\alpha}$ for all $t \in L$. Therefore, any solution starting in $\{x \in B_\lambda \mid \beta_2(x) \le \lambda\}$ stays in $\psi_{t,\alpha}$ and consequently in $\{x \in B_\lambda \mid \beta_1(1) \le \alpha\}$, for all $t \in L$, since $\dot{V}(t,x) \le V(t_0, x_0)$ for all $t \in L$ . For the given conditions, there exists class $\kappa$ functions, $\gamma_1$ and $\gamma_2$ defined on $[0, \lambda]$ such that

$$\gamma_1 \parallel x \parallel \le \beta_1(x) \le V(t,x) \le \beta_2(x) \le \gamma_2 \parallel x \parallel \tag{17}$$

Combining the inequalities defined in Eq. (17), it can be shown that it is seen that

$$\parallel x(t) \parallel \le \gamma_1^{-1} V(t, x(t) \le \gamma_1^{-1} V(t_0, x_0) \le \gamma_1^{-1}\gamma_2 \parallel x(t_0) \parallel$$

Since $\gamma_1^{-1} \circ \gamma_2$ is a class $\kappa$ function , the inequality $\parallel x \parallel \le \gamma_1^{-1}\gamma_2 \parallel x(t_0) \parallel$ shows that the origin is uniformly stable [24].

Now, the proof of the prescribed arbitrary time strong stability can be presented. Let us assume that $V(t,x)$ satisfies the differential inequality defined by Eq. (15) for the initial condition, $V(0, x_0) = V(t_0, x_0)$. Assume that $V(t)$ is the solution of the following differential equation

$$\frac{dV}{dt} = \frac{-\eta V}{t_f - t} \tag{18}$$

Integrating Eq. (18) from $0 \to t$, yields

$$\int_0^t \frac{dV}{V} = \int_0^t \frac{-\eta dt}{t_f - t}$$

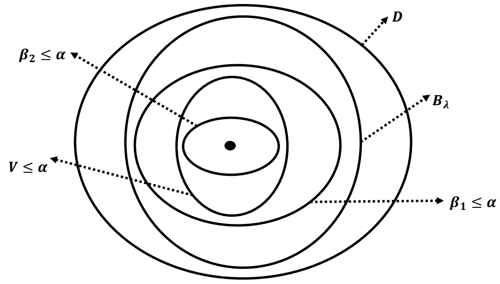


**Fig. 1 Geometric representation of sets in proof of proposition 1.**

$$ln(\frac{V}{V_0}) = \eta ln(\frac{t_f - t}{t_f})$$

where $\eta \geq 1 \in \mathbb{R}$. The solution of Eq. (18) becomes

$$V(t) = V_0(1 - \frac{t}{t_f})^{\eta} \tag{19}$$

where $V_0 = V(t_0)$. From Eq. (19), we have

$$\dot{V}(t) = -\frac{\eta V_0}{t_f}(1 - \frac{t}{t_f})^{\eta - 1} \tag{20}$$

From Eq. (20), it can be seen that $\dot{V}$ dynamics yields to be zero at $t = t_f$ and from Eq. (19), $V(t) = 0$ at $t = t_f$; hence, $\forall t \geq t_f, V(t) = 0$ is maintained.This proves the strong prescribed arbitrary time stability for $z$. Therefore, $T_a = t_f - t_0 = T_f$. The following Lemma 1 is considered for showing the weak prescribed time stability of $V(t, x)$.
*Lemma 1(Comparison Lemma):* Consider the following scalar differential equation.

$$\dot{V_1} = f(t, V1), V_1(t_0) = V_{10}$$

where $f(t, V_1)$ is continuous in $t$ and locally Lipschitz in $V_1$ for all $t \geq 0$ and all $V_1 \in J \subset R$. Let $[t_0, T)$ ($T$ could be infinity) be the maximal interval of existence of solution of $V_1(t)$ , and suppose $V_1(t) \in J$ for all $[t_0, T)$. Let $V_2(t)$ be a continuous function whose upper right-hand derivative $D^+V_2(t)$ satisfies the differential inequality

$$D^+V_2(t) \leq f(t, V_1), V_2(t_0) \leq V_{10}$$

with $V_2(t) \in J$ for all $[t_0, T)$. Then, $V_2(t) \leq V_1(t)$ for all $t \in [t_0, T)$ [24].
Now based on the above Lemma 1, the weakly prescribed time stability can be proven. Let us consider the scalar differential equation given by Eq. (18) and the solution of Eq. (18) is given by Eq. (19). And, the rate of change of Eq. (19) is given by Eq. (20) . It was shown that $V(t)$ and $\dot{V}(t)$ go to zero as $t \longrightarrow t_f$. Now, if we consider the weak stability criteria $\dot{V} \leq \frac{-\eta V}{t_f - t}$, From Comparison Lemma 1, it can be concluded that the solution of $V(t)$ and its derivative will satisfy the inequalities

$$V(t) \leq V_0(1 - \frac{t}{t_f})^{\eta} \tag{21}$$

and

$$\dot{V}(t) \leq -\frac{\eta V_0}{t_f}(1 - \frac{t}{t_f})^{\eta - 1} \tag{22}$$

From Eqs. (21) and (22), it can be concluded that $V(t)$ and $\dot{V}(t)$ go to zero as $t \longrightarrow t_f$. Therefore, the weak prescribed arbitrary time stability of $V(t, x)$ has been proven which results into $T_f \leq t_f - t_0$. The completes the proof of the Proposition 1. Based on the above Proposition 1, the control law has been designed and shown in the following Section 4. The differential inequality given by Eq. (15) nicely fit into the backstepping framework which is shown later.

# V. Control Design

The following section describes the backstepping framework for the proposed linear differential inequality Eq. 15. Explicit derivation for the prescribed arbitrary time controller is given for the first and second order systems, followed by the prescribed time stabilizing nature of the same linear differential inequality Eq. 15 is proven for $n^{th}$order system by using the arguments based on the principle of mathematical induction.

## A. First order system

Consider the following first order system

$$\dot{x} = u; x(t_0) = x_0 \tag{23}$$

where $x \in \mathbb{R}$ is the state of the system, $u \in R_+ \cup \{0\} \times R \to R$ is control, $x = 0$ is the origin of Eq. (23), and $t_0 \in \mathbb{R}_+ \cup \{0\}$ is the initial time. The control law is designed as

$$u = \begin{cases} \frac{-\eta x}{t_f - t} & t_0 \leq t < t_f \\ 0 & otherwise \end{cases} \tag{24}$$

Let the Lyapunov function for the system given by Eq. (23)

$$V = \frac{1}{2}x^2 \tag{25}$$

And, the time derivative of the Lyapunov function gets reduced to

$$\dot{V} = x\dot{x} = \frac{-\eta x^2}{t_f - t} = \frac{-2\eta V}{t_f - t} \tag{26}$$

Based on the Proposition 1, the Eq. (26) has the same structure as that of the dynamics of strongly prescribed arbitrary time stable system. Hence, it can be easily shown that $x = 0$ at $t = t_f$. Again, from Eqs. (23) and (24), $\dot{x} = 0$ for $t \geq t_f$. Hence, Eq. (23) is prescribed arbitrary time stable system with the control law given by Eq. (24).

*Remark 3*

The origin of differential Eq. (18) is attributed to the exponential decay equation, $\dot{x} = -kx$, where $k$ is a constant. If $k$ is changed to $k(t) = \eta/(t_f - t)$ as a time-varying gain, the solution of the resulting non-autonomous equation ($\dot{x} = -k(t)x$) becomes prescribed time convergent.

*Remark 4*

Since the solution of Eqs. (17) and (18) are same, from Eqs. (19) and (20), the expression $|\dot{x}| = \frac{\eta x_0}{t_f}(1 - \frac{t}{t_f})^{\eta-1}$ is always bounded and it can be easily seen that $|\dot{x}| \leq \frac{\eta|x_0|}{t_f}$ for $\forall t \in [0, t_f)$. The maximum value of the control input ($|\dot{x}| = |u|$) occurs at the initial point when the system is at maximum distance from the equilibrium.

*Remark 5*

The control input defined in Eq. (24) grows linearly with respect to the state of system; whereas, in [16], input grows exponentially when state $x < 0$ because $\dot{x} = \frac{-\eta(1-e^{-x})}{t_f - t}$, which implies $|\dot{x}| \leq \frac{\eta|1-e^{-x_0}|}{t_f}$. Clearly, for the same initial conditions (i.e., $x_0 < 0$, $\eta$ and $t_f$) considered in [16], the demanded control input is comparatively high in magnitude in [16] compared to the current findings in this paper. The figure (2) shows the simulation of a first order system with the initial conditions $x(0) = 5$, $\eta = 2$ and $t_f = 7$. It can be seen from Fig. (2) that maximum value of the demanded control input is relatively high in magnitude (dotted line) using the method considered in [16] compared to the proposed approach (solid line) in this paper. Also, using the present approach, the system trajectory follows a smooth convergence to the origin; whereas, in [16], though the states follows a smooth path until time $t_f$, but there is a steep convergence for $t > t_f$. The abrupt change of state also corresponds to a highest demanded control input.

Also, as discussed by [21], the control input for the approach in [16] becomes very large when $x(0) < 0$. This condition is shown in Fig. (3) for $x(0) = -3$, $\eta = 2$ and $t_f = 7$. It can be seen that with the proposed concept, though the convergence scenarios more or less are similar but the requirement of control input is relatively less with the proposed method.

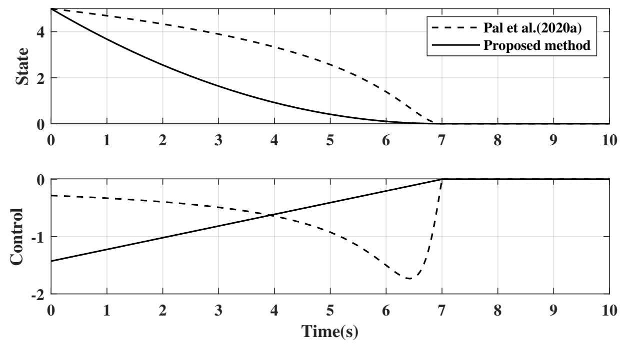


**Fig. 2 First order system trajectory.**

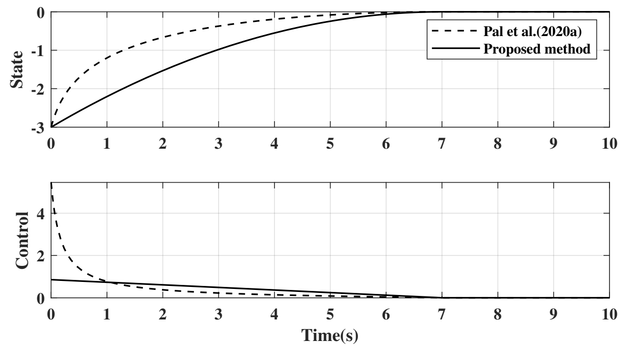


**Fig. 3 First order system trajectory for negative initial condition.**

### B. Second order system

In this sub-section, we extend the concept of prescribed time control to the second order system. Consider the following second order system

$$\dot{x_1} = x_2, \dot{x_2} = u \tag{27}$$

Let $x_{2d}$ be the desired value of $x_2$, then based on the Eq. (24),$x_{2d}$ can be written as

$$x_{2d} = \frac{-\eta_1 x_1}{t_f - t} = -\phi_1, \eta_1 \geq 1 \tag{28}$$

Let $\phi_1 = (\eta_1 x_1)/(t_f - t)$, to backstep, let the change of variable denoted by $w_2$ as

$$w_2 = x_2 - x_{2d} = x_2 + \phi_1 \tag{29}$$

Taking the time derivative of Eq. (29), we have

$$\dot{w_2} = \dot{x_2} + \frac{\partial \phi_1}{\partial x_1}\dot{x_1} + \frac{\partial \phi_1}{\partial t} \tag{30}$$

So, the transformed system becomes

$$\dot{x_1} = w_2 - \phi_1, \dot{w_2} = u + \frac{\partial \phi_1}{\partial x_1}\dot{x_1} + \frac{\partial \phi_1}{\partial t} \tag{31}$$

Let the Lyapunov function be

$$V = \frac{1}{2}(x_1^2 + w_2^2) \tag{32}$$

then, on differentiating Eq. (32), yields

$$\dot{V} = x_1\dot{x_1} + w_2\dot{w_2} \tag{33}$$

Substituting Eqs. (30) and (31) in Eq. (33),

$$\dot{V} = x_1(w_2 - \phi_1) + w_2(u + \frac{\partial \phi_1}{\partial x_1}(w_2 - \phi_1) + \frac{\partial \phi_1}{\partial t}) \tag{34}$$

Now, the control law is taken as

$$u = \begin{cases} -x_1 - \frac{\partial \phi_1}{\partial x_1}(w_2 - \phi_1) - \frac{\partial \phi_1}{\partial t} - \phi_2 & t_0 \le t < t_f \\ 0 & otherwise \end{cases} \tag{35}$$

where

$$\phi_2 = \frac{\eta_2 w_2}{t_f - t}$$

Using the Eqs. (28) and (35), Eq. (34) becomes

$$\begin{aligned} \dot{V} &= -x_1\phi_1 - x_2\phi_2 \\ &= -\frac{\eta_1 x_1^2}{t_f - t} - \frac{\eta_2 x_2^2}{t_f - t} \end{aligned} \tag{36}$$

Let $\eta_2 > \eta_1$ or $\eta_2 = \eta_1 + \varepsilon$, where $\varepsilon > 0$ is a small positive number, then Eq. (36) can be written as

$$\dot{V} = -\frac{\eta_1 x_1^2}{t_f - t} - \frac{\eta_1 w_2^2}{t_f - t} - \frac{\varepsilon w_2^2}{t_f - t} \tag{37}$$

Since $\frac{\varepsilon w_2^2}{t_f - t} \ge 0$, Eq. (37) becomes

$$\dot{V} \le -\frac{\eta_1(x_1^2 + w_2^2)}{t_f - t} \tag{38}$$

In the right side of Eq. (38), the expression $\left(x_1^2 + w_2^2\right)$ can be replaced by $2V$ Eq. (32) and the Eq. (38) gets reduced to

$$\dot{V} \le -\frac{2\eta_1 V}{t_f - t} \tag{39}$$

Now, according to the Proposition 1, it can be said that Eq. (39) is the dynamics of prescribed arbitrary time weak stable system. Hence, by the similar arguments made for the first order system (sub-section 4.1), it can be concluded that $V = 0$ at $t \le t_f$. Further, the control law defined by Eq. (35) takes the system states to $x_1 = 0$ and $w_2 = 0$ at $t \le t_f$. In the following, the prescribed time control can be proved for $n + 1^{th}$ order system. It is assumed that there is a control input denoted by $u_n$ can stabilize the system of order $n$ in prescribed arbitrary time. By inspecting Eqs. (25) and (32), the Lyapunov function can be constructed for an $n + 1^{th}$order system as

$$V_{n+1} = V_n + \frac{1}{2} w_{n+1}^2 \tag{40}$$

where $V_n$ and $V_{n+1}$ are the Lyapunov functions for the systems of order $n$ and $n + 1$, respectively. Taking the time derivative of Eq. (40), yields

$$\dot{V}_{n+1} = \frac{\partial V_n}{\partial x_1}\dot{x_1} + \frac{\partial V_n}{\partial w_2}\dot{w_2} + ... + \frac{\partial V_n}{\partial w_n}\dot{w}_n + w_{n+1}\dot{w}_{n+1} \tag{41}$$

After simplification, Eq. (41) becomes

$$\dot{V}_{n+1} = x_1\dot{x_1} + w_2\dot{w_2} + ... + w_n\dot{w}_n + w_{n+1}\dot{w}_{n+1} \tag{42}$$

Let $u_{n+1}$ be the control for $n + 1^{th}$ order system, then using the change of variable defined in Eq. (29), Eq. (42) can be written as

$$\begin{aligned} \dot{V}_{n+1} &= -x_1\phi_1 - w_2\phi_2 ... - w_n\phi_n - w_{n+1}\dot{w}_{n+1} \\ &= -x_1\phi_1 - w_2\phi_2 ... - w_n\phi_n - w_{n+1}(u_{n+1} - \dot{x}_{(n+1)d}) \end{aligned} \tag{43}$$

where

$$\phi_n = \frac{\eta_n w_n}{t_f - t}, \eta_n \geq 1, n \geq 2 \tag{44}$$

In equation (43), $u_{n+1}$ can be derived for the $n+1^{th}$ order system by following the backstepping procedure described in sections 4.1 and 4.2 for the first and second order systems, respectively, and given here in abstract form as

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\dots \\
\dot{x}_n &= x_{n+1}
\end{aligned}$$

$$\dot{x}_{n+1} = u_{n+1} \tag{45}$$

Using Eq. (29), $u_{n+1}$ can be expressed as

$$u_{n+1} = -x_n + \dot{x}_{(n+1)d} \tag{46}$$

where $x_{(n+1)d}$ acts as a virtual control and stabilizes the system up to order $n$. The expression $x_{(n+1)d}$ is related to the previous states in the following means

$$x_{(n+1)d} = f(x_1, x_2, ...x_n)$$

or,

$$\dot{x}_{(n+1)d} = \frac{\partial f}{\partial x_1}\dot{x}_1 + \frac{\partial f}{\partial x_2}\dot{x}_2 + ...... + \frac{\partial f}{\partial x_n}\dot{x}_n + \frac{\partial f}{\partial t}$$

Now, substituting the expressions of $u_{n+1}$ ,$x_{n+1}, \dot{x}_{(n+1)d}$ and $\phi_n$ in Eq. (43) and using the definition of quadratic Lyapunov function defined in Eq. (40) , expression of $\dot{V}_{n+1}$ in Eq. (43) gets reduced

$$\dot{V}_{n+1} \leq -\frac{2\eta_{min} V}{t_f - t}, \eta_{min} = min(\eta_i | i = 1, 2, 3...n) \tag{47}$$

By invoking Proposition 1, it can be concluded that $n^{th}$ order system becomes prescribed arbitrary time stable. And, by mathematical induction, the above result holds for any arbitrary $n \in \mathbb{N}$. This completes the proof that $n^{th}$ order system is prescribed arbitrary time stabilizable with the control $u_n$.

### C. Boundedness of $u(t)$

In this sub-section, the boundedness of the proposed control given by Eq. (35) is shown. Let $u_{max}$ be the maximum required control, then from Eq. (35), it can be seen that $u = u_{max}$ will occur at $t \leq t_f$. The first part of Eq. (35) is rewritten as

$$u = -x_1 - \frac{\partial \phi_1}{\partial x_1}(w_2 - \phi_1) - \frac{\partial \phi_1}{\partial t} - \phi_2 \tag{48}$$

Now, using the definitions of $\phi_1 = (\eta_1 x_1)/(t_f - t)$ and $\phi_2 = (\eta_2 w_2)/(t_f - t)$, $\partial\phi_1/\partial x_1$ and $\partial\phi_1/\partial t$ are obtained as

$$\frac{\partial \phi_1}{\partial x_1} = \frac{\eta_1}{t_f - t} \tag{49}$$

$$\frac{\partial \phi_1}{\partial t} = \frac{\eta_1 x_1}{(t_f - t)^2} \tag{50}$$

Using Eqs. (48), (49) , (50) and (29), Eq. (48) becomes

$$u = -x_1 - \frac{\eta_1 x_2}{t_f - t} - \frac{\eta_1 x_1}{(t_f - t)^2} - \frac{\eta_2 (x_2 + \frac{\eta_1 x_1}{t_f - t})}{t_f - t}$$
$$= -x_1 - \frac{(\eta_1 + \eta_2) x_2 + \eta_1 \eta_2 x_1}{t_f - t} - \frac{\eta_1 x_1}{(t_f - t)^2} \tag{51}$$

Now, the boundedness of control torque $u$ will be proven using the extreme value theorem. If a real-valued function $u(t)$ is continuous on a closed interval $t \in [a, b]$, then $u$ must attain a maximum and minimum each at least once. As a result, $u(t)$ will remain bounded on that interval. Now, the control input $u(t)$ defined in Eq. (51) on the interval $[0, t_f)$ can be written as

$$u(0) = -x_1(0) - \frac{(\eta_1 + \eta_2) x_2(0) + \eta_1 \eta_2 x_1(0)}{t_f} - \frac{\eta_1 x_1(0)}{t_f^2} \tag{52}$$

Let us assume that $\eta_1 \geq 1$ and $\eta_2 \geq 1$ are sufficiently large such that $x_1 \to 0, x_2 \to 0$, as $t \to t_f$. For $t \to t_f$, the term $u(t_f)$ in Eq. (51) goes to infinity, i.e., $u(t_f) \to \infty$; but both the numerator and the denominator in $2^{nd}$ and $3^{rd}$ terms (denoted by $T_2$ and $T_3$) of Eq. (51) tends to zero, making it a situation of undefined. The limits of these terms can be estimated using L'Hopital's rule as following. Let us assume $\lim_{t \to t_f} u(t) = \lim_{t \to t_f} \dot{x}_2(t) = L$, then

$$T_2 = \frac{\frac{d}{dt}\left[(\eta_1 + \eta_2) x_2(t_f) + \eta_1 \eta_2 x_1(t_f)\right]}{\frac{d}{dt}\left[t_f - t\right]}$$
$$= -(\eta_1 + \eta_2)\dot{x}_2(t_f) - \eta_1 \eta_2 \dot{x}_1(t_f)$$
$$= -(\eta_1 + \eta_2) L - \eta_1 \eta_2 x_2 \tag{53}$$

$$T_3 = \frac{\frac{d}{dt}\left[\eta_1 x_1(t_f)\right]}{\frac{d}{dt}\left[(t_f - t)^2\right]} = -\frac{\eta_1 \dot{x}_1(t_f)}{2} = -\frac{\eta_1 x_2(t_f)}{2} \tag{54}$$

Using the above terms for $t \to t_f$, Eq. (51) can be simplified as

$$L = -x_1(t_f) - T_2 - T_3$$
$$= -x_1(t_f) + (\eta_1 + \eta_2) L + \eta_1 \eta_2 x_2(t_f) + \frac{\eta_1 x_2(t_f)}{2}$$
$$= \frac{-x_1(t_f) + \eta_1 \eta_2 x_2(t_f) + \frac{\eta_1 x_2(t_f)}{2}}{1 - \eta_1 - \eta_2} \tag{55}$$

From Eqs. (52) and (55), it is clear that the values of $u(0)$ and $u(t_f) \to L$ are finite value; hence, $u(t)$ is continuous on the interval $[0, t_f)$. From the above analysis, it can be stated that using the extreme value theorem, $u(t)$ is bounded on the interval $[0, t_f)$ under the assumptions $\eta_1 \geq 1$ and $\eta_2 \geq 1$ such that $x_1 \to 0, x_2 \to 0$, as $t \to t_f$ using the control law defined by Eq. (51). Now, the effect of $t_f$ will be examined on $u(t)$. Let us consider the initial conditions for the system given by Eq. (27) are $x_1(0) = -0.1$, $x_2(0) = 0.1$ and the control gains are $\eta_1 = \eta_2 = 2$. The control requirement is shown in Fig. (4) at $t = 0$. It can be stated that the control requirement drastically reduces if we relax the desired convergence time. Note that if $t_f \to \infty$, the control becomes $|u| \leq |-x_1(0)|$ Eq. (52) and structurally looks like a simple proportional feedback control with unity gain. In the next section, the application of the proposed control is extended for the attitude control of a satellite.

### D. Spacecraft Control with no disturbance

In this section, the application of prescribed time arbitrary control algorithm is discussed on the attitude dynamics of a rigid spacecraft. For designing the proposed control, complete knowledge of system dynamics is required. The derivation of the controller for satellite attitude control without disturbance is described in this part, and the disturbance term is incorporated in the dynamics in the next section.

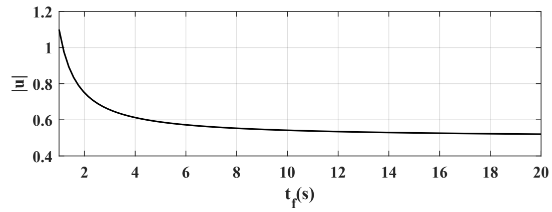


**Fig. 4 Control at initial time $u(0)$ $vs.$ desired convergence time $t_f$.**

***Proposition 2:*** If the error dynamics in Eq. (58) is considered with $d(t) = 0$, a control command $u \in \mathbb{R}^3$ can be designed for attitude stabilization and tracking control such that for any initial condition $q(0)$ and $\omega(0)$, the quaternion error approaches to unit quaternion $q = [1, 0, 0, 0]^T$ and error in angular velocity approaches to zero $\omega_e = \omega - \omega_d = [0, 0, 0]^T$ within desired time $t_f$. The proposed control is given by

$$u = \begin{cases} J(2G(-\tilde{q}_e - Pq_w - \dot{\phi} - \mu) + \dot{\omega}_d^B) + \hat{\omega}J\omega, & t_0 \le t < t_f \\ 2J\dot{\omega}_d^B + \hat{\omega}J\omega, & otherwise \end{cases} \tag{56}$$

where $\phi = [\phi_1, \phi_2, \phi_3, \phi_4]^T \in \mathbb{R}^4$ and $\mu = [\mu_1, \mu_2, \mu_3, \mu_4]^T \in \mathbb{R}^4$ or

$$\phi_i = \frac{\eta_i \tilde{q}_e}{t_f - t}, \mu_i = \frac{\eta_j (q_{wi} + \phi_i)}{t_f - t}$$

where $i = 1, 2, 3, 4$; $j = 5, 6, 7, 8$, $\eta_i, \eta_j \ge 1$, $\tilde{q}_e = q_e - q_I$, $q_I = [1, 0, 0, 0]^T$ is a unit quaternion and $P \in \mathbb{R}^{4\times 4}$ with $a, b = \{1, 2, 3, 4\}$ is defined as

$$P = \left[\frac{\partial \phi_a}{\partial \tilde{q}_{eb}}\right] = \begin{bmatrix} \frac{\partial \phi_1}{\partial \tilde{q}_{e1}} & 0 & 0 & 0 \\ 0 & \frac{\partial \phi_2}{\partial \tilde{q}_{e2}} & 0 & 0 \\ 0 & 0 & \frac{\partial \phi_3}{\partial \tilde{q}_{e3}} & 0 \\ 0 & 0 & 0 & \frac{\partial \phi_4}{\partial \tilde{q}_{e4}} \end{bmatrix} \tag{57}$$

Proof: The system given by Eq. (10) can be written in a cascaded form as follows

$$\dot{q}_e = q_w, \dot{q}_w = u_1 \tag{58}$$

where

$$u_1 = \frac{1}{2} G^T (\dot{\omega} - \dot{\omega}_d^B)$$

$$q_w = \frac{1}{2} q_e \otimes (\boldsymbol{\omega} - \boldsymbol{\omega}_d^B)$$

where $u_1$ is a prescribed arbitrary time controller which will be designed later in this section. Using Eq. (2), $u_1$ can be written as following

$$u_1 = \frac{1}{2} G^T (J^{-1}(u - \hat{\omega}J\omega) + \dot{\omega}_d^B) \tag{59}$$

Backstepping approach is used to derive the control torque $u$ which will ensure the convergence of the quaternion vector part and error in angular velocity to zero in desired time $t_f$. To backstep, define another variable as

$$\tilde{q}_w = q_w - q_{wd} \tag{60}$$

where $q_{wd}$ is the desired value of $q_w$ which acts as virtual control and according to the Eq. (29) defined as

$$q_{wd} = -\phi(\tilde{q}_{e1}, \tilde{q}_{e2}, \tilde{q}_{e3}, \tilde{q}_{e4}) = -\phi \tag{61}$$

where $\phi$ is defined as follows

$$\phi = \left[\frac{\eta_1 \tilde{q}_{e1}}{t_f - t}, \frac{\eta_2 \tilde{q}_{e2}}{t_f - t}, \frac{\eta_1 \tilde{q}_{e3}}{t_f - t}, \frac{\eta_1 \tilde{q}_{e4}}{t_f - t}\right]^T \tag{62}$$

Now, Eq. (60) becomes

$$\tilde{q}_w = q_w + \phi \tag{63}$$

Taking the time derivative of Eq. (63), yields

$$\dot{\tilde{q}}_w = \dot{q}_w + \dot{\phi} + P\dot{\tilde{q}}_e \tag{64}$$

where $P$ represents $\frac{\partial \phi}{\partial \tilde{q}_e}$ as described in Eq. (57). Now, the Eq. (58) using the above transformations becomes

$$\dot{\tilde{q}}_e = \tilde{q}_w - \phi, \dot{\tilde{q}}_w = u_1 + \dot{\phi} + P\dot{\tilde{q}}_e \tag{65}$$

It can be noted in Eq. (65) that $\dot{\tilde{q}}_e = \dot{q}_e$ since $\tilde{q}_e = q_e - q_I$. Now, consider the following Lyapunov function

$$V = \frac{1}{2}\tilde{q}_e^T \tilde{q}_e + \frac{1}{2}\tilde{q}_w^T \tilde{q}_w \tag{66}$$

Taking the time derivative o f $V$, yields

$$\begin{aligned} \dot{V} &= \tilde{q}_e^T \dot{\tilde{q}}_e + \tilde{q}_w^T \dot{\tilde{q}}_w \\ &= \tilde{q}_e^T (\tilde{q}_w - \phi) + \tilde{q}_w^T (u_1 + \dot{\phi} + P\dot{\tilde{q}}_e) \\ &= \tilde{q}_e^T (\tilde{q}_w - \phi) + \tilde{q}_w^T (u_1 + \dot{\phi} + P q_w) \end{aligned} \tag{67}$$

Let the control be

$$u_1 = \begin{cases} -\tilde{q}_e - \dot{\phi} - P q_w - \mu(\tilde{q}_w) & t_0 \le t < t_f \\ 0 & otherwise \end{cases} \tag{68}$$

where

$$\mu = \left[\frac{\eta_1 \tilde{q}_{w1}}{t_f - t}, \frac{\eta_2 \tilde{q}_{w2}}{t_f - t}, \frac{\eta_3 \tilde{q}_{w3}}{t_f - t}, \frac{\eta_4 \tilde{q}_{w4}}{t_f - t}\right]^T \tag{69}$$

After substituting the value of $u_1$ into Eq. (67) and using the relation $\tilde{q}_e^T \tilde{q_w} = \tilde{q_w}^T q_e$, we get

$$\dot{V} = -\tilde{q}_e^T \phi - \tilde{q}_w^T \mu \tag{70}$$

Using the Eqs. (62),(69), Eq. (70) can be written as

$$\dot{V} = -\sum_{k=1}^{4} \frac{\eta_k \tilde{q}_{ek}^2}{t_f - t} - \sum_{k=1}^{4} \frac{\eta_{k+4} \tilde{q}_{wk}^2}{t_f - t} \tag{71}$$

Let us assume $\eta_{min} = \eta_1$, one of the $\eta_i$ $(i = 1, 2, 3...8)$ will be minimum so either of the 8 coefficients can be taken as minimum for the following proof. Now, for a small quantity $\epsilon_j \ge 0$, the following holds

$$\eta_{j+1} = \eta_{min} + \epsilon_j, 1 \le j \le 7$$

Substituting the above coefficients in Eq. (71), we get

$$\dot{V} = -\eta_{min} \sum_{k=1}^{4} \frac{1}{t_f - t} \left[\tilde{q}_{ek}^2 + \tilde{q}_{wk}^2\right] - \left[\sum_{k=2}^{3} \frac{\epsilon_k \tilde{q}_{ek}^2}{t_f - t} + \sum_{k=1}^{4} \frac{\epsilon_{k+3} q_{wk}^2}{t_f - t}\right] \tag{72}$$

Let us denote

$$\lambda = \sum_{k=2}^{3} \frac{\epsilon_k \tilde{q}_{ek}^2}{t_f - t} + \sum_{k=1}^{4} \frac{\epsilon_{k+3} q_{wk}^2}{t_f - t} \tag{73}$$

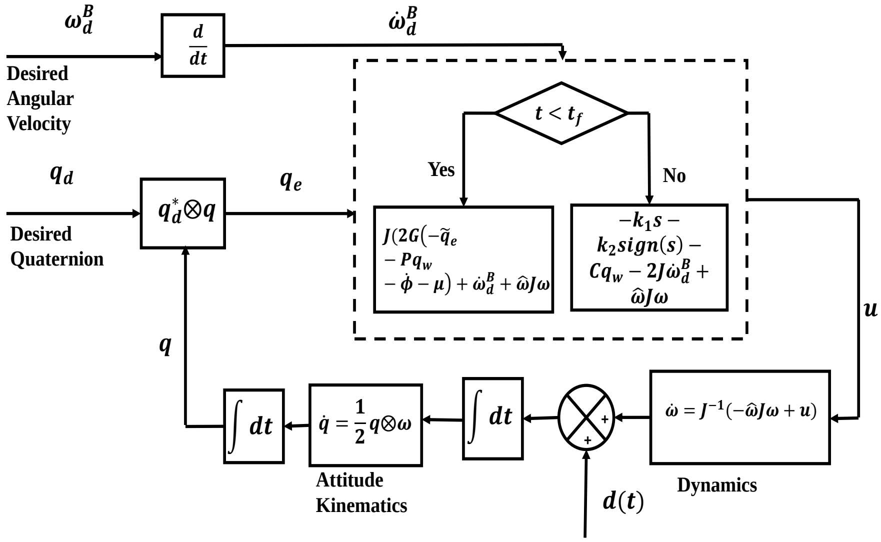


**Fig. 5 Block diagram for control architecture**

According to the Eqs. (37) and (38), $\lambda \geq 0$. And, the term in Eq. (72) can be written as

$$\sum_{k=1}^{4} \left[\tilde{q}_{ek}^2 + q_{wk}^2\right] = \tilde{q}_e^T \tilde{q}_e + \tilde{q}_w^T \tilde{q}_w \tag{74}$$

Using the Eqs. (73) and (74), the Eq. (72) gets reduced

$$\begin{aligned} \dot{V} &\leq -\frac{\eta_{min}(\tilde{q}_e^T \tilde{q}_e + \tilde{q}_w^T \tilde{q}_w)}{t_f - t} \\ &\leq -\frac{2\eta_{min} V}{t_f - t} \end{aligned} \tag{75}$$

Hence $\dot{V} \leq 0$ and therefore, as stated in proposition 1, attitude error and error in angular velocity approaches to zero as $t \longrightarrow t_f$. It can be noted that the control law $u$ can be found out by substituting virtual control $u_1$ derived from (68) into (59).

In the next section, we design the robust control for $t \geq t_f$, after the system reaches to the equilibrium. Before $t < t_f$, the time-varying gain can cancel the disturbances affecting the system dynamics. But, after $t \geq t_f$, only desired attitude derivative and nonlinear terms in spacecraft dynamics are canceled. Therefore, it is necessary to provide robustness against disturbances to the presented control law.

### E. Robustness Analysis

In order to cancel bounded disturbances, a sliding mode controller is added to the backstepping framework, making the combined controller a robust controller in nature. An exponential reaching law based sliding mode controller is used to cancel the disturbance. With considering the disturbance in the system defined by Eq. (58), yields

$$\dot{q}_e = q_w, \dot{q}_w = u_r + T_d \tag{76}$$

where $T_d = \frac{1}{2} G^T J^{-1} d(t)$ is a Lipschitz continuous signal and $u_r$ is a robust control which composed of $u_1$ (prescribed arbitrary time control) and sliding mode control (denoted by $u_{SMC}$) that helps the controller to reject the disturbance. It is assumed that $T_d$ is bounded by $\parallel T_d \parallel \leq \tau_d$, where $\tau_d$ is the bound of the disturbance. The $u_r$ can be written as

$$u_r = u_1 + k u_{SMC} \tag{77}$$

where $k$ is defined as follows:

$$k = \begin{cases} 0 & t_0 \leq t \leq t_f \\ 1 & otherwise \end{cases} \tag{78}$$

In equation (78), $k$ ensures that sliding mode control is only active after the desired convergence time, $t_f$ and before which prescribed arbitrary time control ($u_1$) can reject the disturbances. This technique helps the system to converge to desired attitude and angular velocities even in the presence of external disturbances. Let us define a sliding surface ($s \in \mathbb{R}^4$) as follows

$$s = q_w + C\tilde{q}_e \tag{79}$$

where $C \in \mathbb{R}^{4\times4}$ is a gain matrix such that $C_{ij} > 0, i, j = 1, 2, 3, 4$. The derivative of $s$ is given by

$$\dot{s} = \dot{q}_w + C\dot{\tilde{q}}_e = \dot{q}_w + Cq_w \tag{80}$$

Using Eqs. (76) and (77), Eq. (80) becomes

$$\dot{s} = u_1 + ku_{SMC} + T_d + Cq_w \tag{81}$$

In Eq. (81), the sliding mode control, $u_{SMC}$, is taken as following based on exponential reaching law.

$$u_{SMC} = -k_1 s - k_2 sign(s) - Cq_w \tag{82}$$

where $k_1, k_2 \in \mathbb{R}^{4\times4}$ are diagonal and positive definite gain matrices. Consider the following Lyapunov function

$$V = \frac{1}{2}s^T s \tag{83}$$

After differentiating *wrt* time of Eq. (83), using the Eq. (81), Eq. (83) becomes

$$\begin{aligned} \dot{V} &= s^T \dot{s} \\ &= s^T(u_1 + ku_{SMC} + T_d + Cq_w) \end{aligned} \tag{84}$$

It is already discussed that after $t = t_f$,$u_1 = 0$ and $k = 1$. Using these conditions and control defined by Eq. (82), Eq. (84) becomes

$$\dot{V} = -k_1 s^T s - k_2|s| + s^T \tau_d \tag{85}$$

If $|k_2| \geq |\tau_d|$, Eq. (85) can be written as

$$\dot{V} \leq -k_1|s^T s| \tag{86}$$

Since, $\dot{V}$ is negative definite for all $t \geq t_f$, it proves that the sliding mode control input, $u_{SMC}$, is also stabilizing. Therefore, it can be concluded that the robust control method presented in this section is globally stable and ensures the convergence of system states in prescribed time. The block diagram representing the overall control architecture is presented in Fig. (5). The figure (5) depicts the switching control strategy clearly by showing that prescribed time control brings the system to equilibrium within time $t_f$ and after $t_f$ the sliding mode based disturbance rejection is activated.

# VI. Simulation

In this section, the results of numerical simulation of prescribed arbitrary time stabilization and tracking of spacecraft with and without external disturbance are presented. Initial conditions are provided in Table 1. Other parameters such as inertia matrix, controller parameters, disturbance magnitude and desired angular velocities are provided in Table 2. It should be noted that initial conditions and all other parameters are kept same throughout the simulations.

1. **Attitude stabilization:** For attitude stabilization, desired quaternion is $q_d = [1, 0, 0, 0]^T$ and desired angular velocity is $\omega_d^D = [0, 0, 0]^T$. The figure (6) shows the results for stabilization without disturbance. It is observed from Fig. (6) that the attitude quaternion vector part $q_v = [q_1, q_2, q_3]^T \in \mathbb{R}^3$ approaches to zero within prescribed settling time, $t_f = 5$ seconds. Angular velocities also approach to zero within desired settling time, $t_f = 5$ seconds. The

| Quantity | Initial condition |
|---|---|
| Quaternion$(q(0))$ | $[0.9981, 0.0262, -0.0237, 0.0506]^T$ |
| Angular velocity(rad/s)$(\omega(0))$ | $[0.2, 0.1, -0.3]^T$ |

**Table 1 Initial conditions for simulation.**

| Parameter | Value |
|---|---|
| $\eta_i, (i = 1, 2, 3...8)$ | 7 |
| $J$ | $diag\{1, 3, 2\}\ Kg.m^2$ |
| $d(t)$ | $[0.001, 0.001, 0.001]^T sin(4\pi t)$ N-m |
| $\omega_d^D$ | $[0.5, 0.5, 0.4] sin(t)$ rad/s |
| $k_1$ | $diag\{2, 2, 2\}$ |
| $k_2$ | $diag\{0.001, 0.001, 0.001\}$ |
| $C$ | $diag\{2, 2, 2\}$ |

**Table 2 Simulation parameters.**

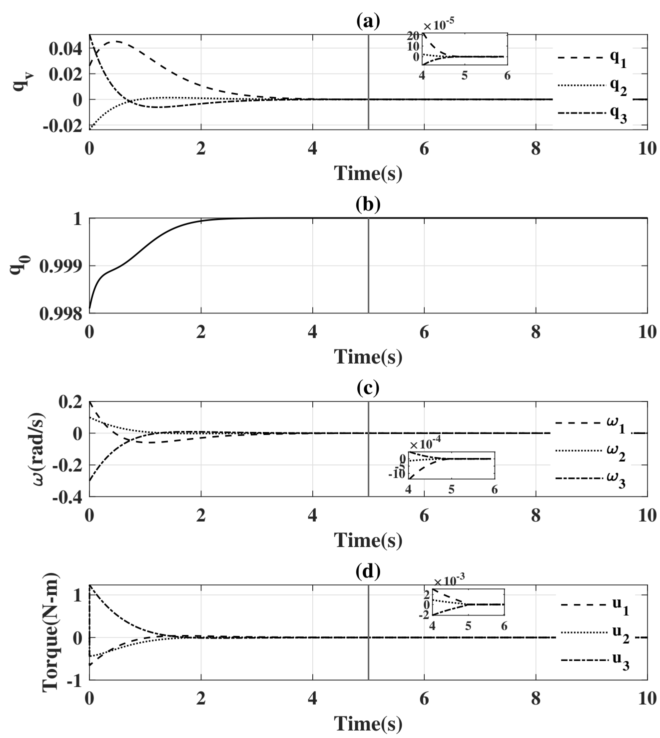


**Fig. 6 Stabilization of spacecraft without disturbance**

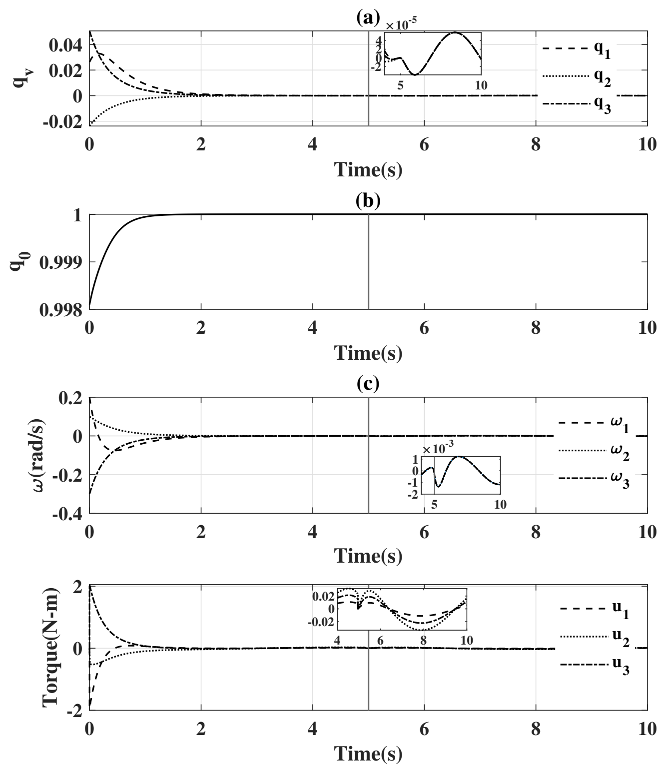


**Fig. 7 Stabilization of spacecraft with disturbance**

torque required for this control scheme is maximum at the initial time because the system is at maximum distance from equilibrium.

Attitude quaternion approaches to unit quaternion within the limit of desired time $t_f$ as shown in Figs. 7(a) and 7(b). The error in angular velocity approaches to zero within the limit of desired time $t_f$ which can be seen in Fig. 7(c). The figure 7(d) demonstrate the torque required for stabilization. The control torque is non-zero after desired settling time, $t_f = 5$ seconds because it is continuously rejecting the disturbance, as seen in Fig. 7(d). As a result, the angular velocity in Fig. 7(c) is non-zero, on the order of 0.001 rad/s.

It can be noticed from the above simulations that there is an abrupt transition at $t_f = 5$ seconds, which indicates that the time varying control gain $(\eta/t - t_f)$ is increasing rapidly in effort to bring the attitude states to origin as $t \rightarrow t_f$. If control availability is high, then this jerky transition can be further reduced by increasing the control gains $\eta$. Increase in control gain $\eta$ corresponds to higher required control torque at $t = 0$.

**2. Attitude tracking:** For tracking, desired quaternion is computed by integrating the derivative of desired quaternion which is obtained from desired angular velocity. The desired angular velocity in body frame is obtained using the equation $\omega_d^B = R(q_e)^T \omega_d^D$ and $\dot{\omega}_d^B$ is obtained using Eq. (11). Figure (8) shows the results for attitude tracking without presence of external disturbance. It is observed that after settling time $t_f = 5$ seconds, output follows the desired attitude without any visible lag or delay. Similar tracking performance is also seen from Fig. (9) in angular velocity tracking. The control torque for tracking is shown in Fig. (10). Maximum control torque demand occurs the initial time when the system is at maximum distance from equilibrium.

Now, the simulation results of spacecraft attitude tracking with the presence of external disturbance are presented. Attitude quaternion tracks the desired quaternion within the limit of desired time $t_f$ as presented in Fig. (11). The actual angular velocity of spacecraft tracks the desired angular velocity as seen in Fig. (12). The demanded torque during tracking maneuver is shown in the Fig. (13). The findings from the current simulation show that the actual convergence time is less than the desired convergence time. If the gains $\eta_i$ are reduced any further, the convergence time begins to grow and approaches $t_f$ and vice-versa. It can be seen that in the absence of external disturbance, the highest torque requirement occurs at the start. When applying this control scheme to actual systems, the torque demand at the beginning point is a key quantity because if the torque requirement exceeds the actuator limit, the control designer may

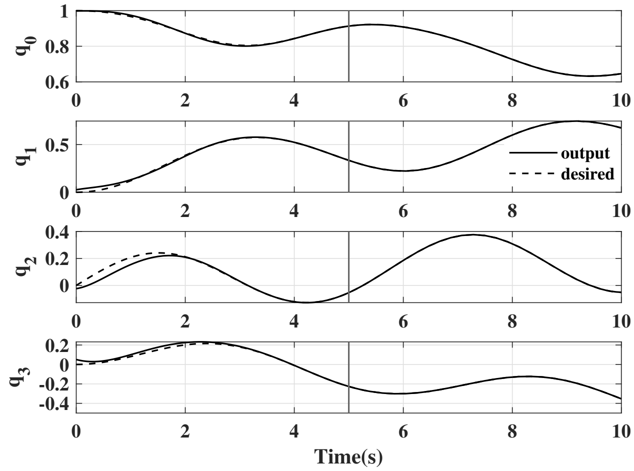


**Fig. 8 Attitude tracking of spacecraft without disturbance**

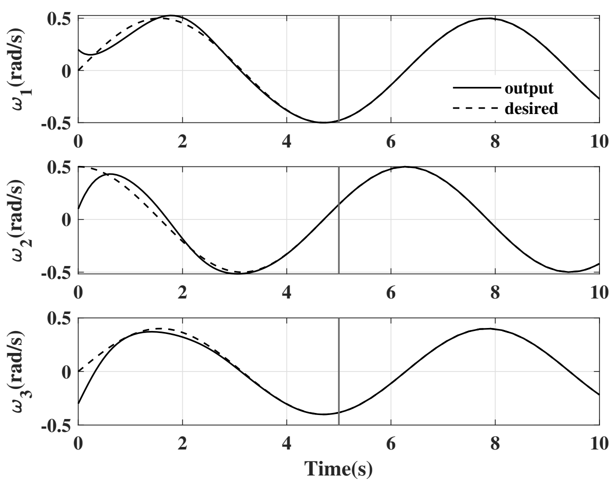


**Fig. 9 Angular velocity tracking of spacecraft without disturbance**

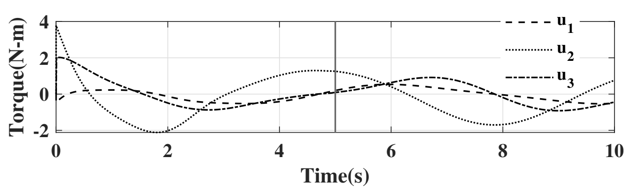


**Fig. 10 Control torque during tracking of spacecraft without disturbance**

have to raise the settling time requirement to reduce the maximum necessary torque.

Additionally, the performance of the presented control algorithm is tested for the inertia matrix, denoted as $J_u$, with off-diagonal terms. The diagonal terms of $J_u$ are increased by 50 percent from that of $J$. The disturbance model remains the same as shown in the Table 1. The off-diagonal inertia matrix for the system is given below.

$$J_u = \begin{bmatrix} 1.5 & 0.5 & 0.4 \\ 0.5 & 4.5 & 0.4 \\ 0.4 & 0.4 & 3 \end{bmatrix} kg.m^2$$

It can be noted that the dynamics propagates using the inertia matrix $J_u$; whereas, the control law uses the value of inertia matrix given by $J$. This implies that we have uncertain inertia matrix $J$ available to us for computing control input and the actual inertia matrix is $J_u$ for the propagating the attitude dynamics. Initial conditions for quaternion attitude and angular velocity, control gains and disturbance magnitude are kept same as provided in Table 1 and Table 2.

The figure 14(a) shows that the system stabilizes within prescribed time, $t_f = 5$ seconds. However, oscillation of the very small magnitude of attitude quaternion around the equilibrium point can be clearly seen. This is due to presence of sinusoidal disturbance in the dynamics. The figure 14(c) shows the convergence of angular rate to zero. Angular velocity also has small amplitude oscillation around the equilibrium point. The figure 14(d) shows the required torque to keep the system on equilibrium. Non-zero torque is always acting on the system to reject the disturbance, which enters the system dynamics both directly via external sources and indirectly via the uncertain inertia matrix.

This is required to mention that the control torque requirement is considerably high in magnitude at the beginning of simulation and increases as the system initial conditions are farther away from equilibrium. If the system is far from equilibrium, the control requirement may become excessive when our objective is to converge the system to equilibrium in a veryshort period of time. The control designer must strike a balance between the settling time and the maximum needed control torque, so that the actuators do not saturate. The plot in Fig. (15) shows the control requirement at $t = 0$, for the attitude stabilization with no disturbance case to demonstrate the effect of $t_f$. Clearly, very short convergence time demands very high control torque and vice-versa.

## VII. Conclusion

For quaternion-based spacecraft attitude control, an algorithm known as "prescribed arbitrary time control" was presented in this study. This research's control scheme ensures prescribed arbitrary time stability and attitude tracking with good performance for arbitrary initial conditions and system parameters. The control law handled the rejection of disturbances quite well. It was shown that the control demand for disturbance rejection rose dramatically, which might be an issue when implementing this control law to actual systems. It is observed that the control demands at the initial time is very high, which can saturate the actuators. There are analytical results presented for a second order system for how convergence time affects the maximum control torque that needs to be applied to the system. Future work on this can include testing of this control algorithm under input saturation. Experimentation would corroborate the reported findings even if the suggested controllers show impressive performance in simulations.

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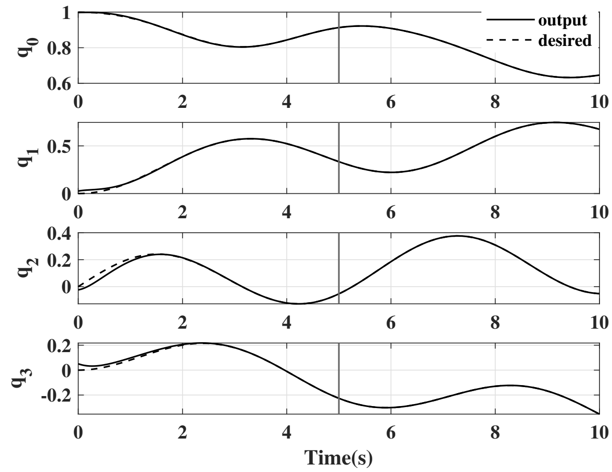


**Fig. 11  Attitude tracking of spacecraft with disturbance**

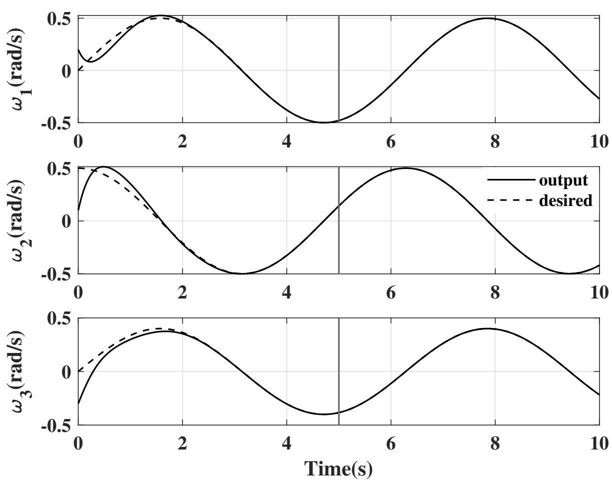


**Fig. 12  Angular velocity tracking of spacecraft with disturbance**

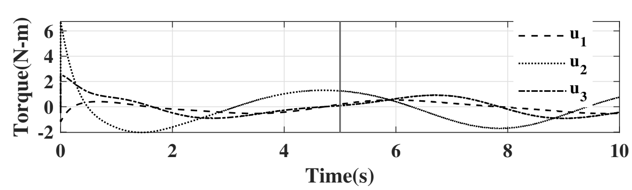


**Fig. 13  Control torque during tracking of spacecraft with disturbance**

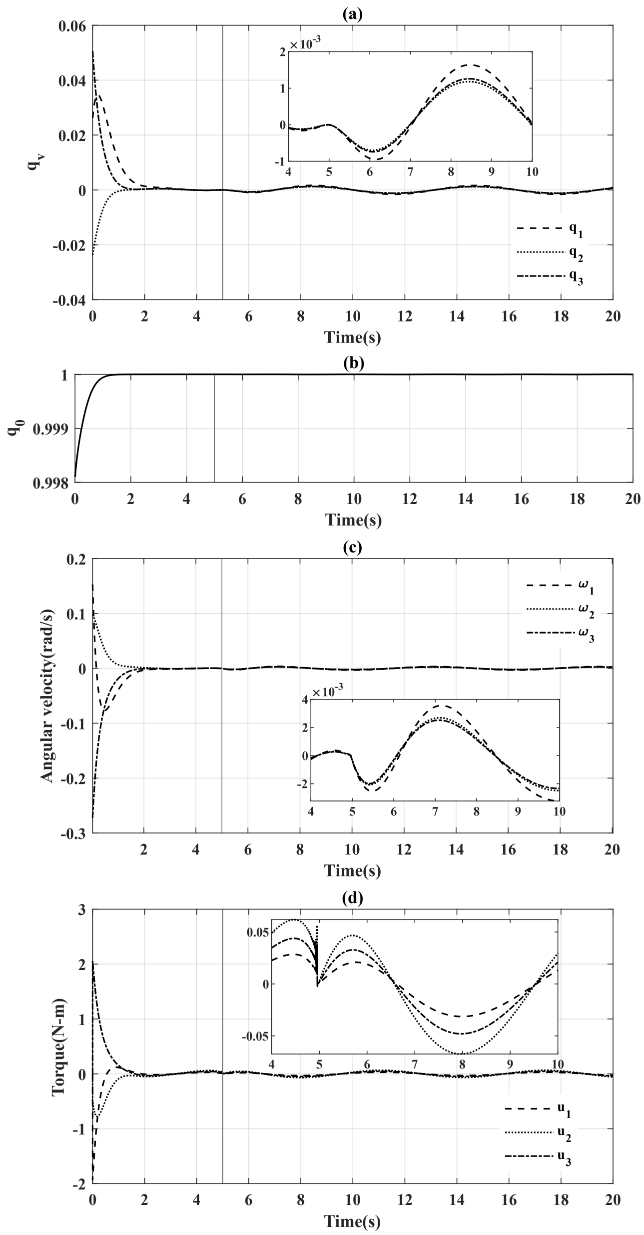


**Fig. 14 Stabilization with uncertain inertia and disturbance**

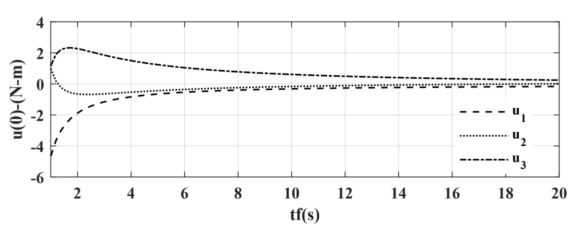


**Fig. 15 Control requirement at the initial moment with varying $t_f$**

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